# ANALYSIS OF CURVED TAPERED BEAM ELEMENTS USING FORCE BASED FORMULATION

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## ABSTRACT

This paper presents a force based finite element model for linear static analysis of 2-D curved tapered Timoshenko beam elements having variable depth and uniform width. The formulation does not use any additional assumption for the variation of displacement. Hence the derived element stiffness matrix is exact even for curved tapered members with variable depth. Displacement based finite element approach that is commonly used in commercial finite element packages, uses an assumption for the distribution of displacement along the element, which does not satisfy the governing differential equations of the Timoshenko element for the case of curved members. Hence, such approaches does not provide closed form exact solutions for the deflection of a curved tapered beam and requires discretization using a large number of elements. The results of a linear static analysis of a 2-D curved tapered cantilever Timoshenko beam obtained using the proposed model were compared with those of discretized displacement based models to highlight the accuracy and computational efficiency of the proposed force based finite element model in analysing 2-D structures with complex geometries

***Keywords:*** force based formulation, Timoshenko beam, curved tapered member

## 1. INTRODUCTION

Curved tapered members, though not used as frequently as prismatic members in design of everyday structures, appear as aesthetically pleasing and structurally sound options when designing structures with complex architecture.

Traditional finite element methods of analysis of structures follow a procedure that utilizes the displacement based finite element approach. Displacement based finite element formulations are based on the assumption of displacement interpolation functions. Hence, analysis of structures composed of curved tapered members requires discretization using a large number of elements.

Force based finite element formulation is based on the selection of force interpolation functions such that equilibrium is satisfied at any internal point along the element, hence is exact even for curved tapered members. Therefore, force based approach offers a more accurate and computationally efficient method of analysis of structures containing curved tapered members.

Force based approach has been used by Spacone et al., 1996 for non-linear analysis of reinforced concrete frames. Linear dynamic analysis of complex structures with curved 3-D members has been performed by Molins et al.[1].

This paper presents a force based model for linear static analysis of 2-D curved tapered Timoshenko beam elements and investigates the accuracy and computational efficiency of the proposed model with discretized displacement based finite element models.

## 2. FORCE BASED FINITE ELEMENT FORMULATION OF 2-D CURVED TAPERED TIMOSHENKO BEAM ELEMENT STIFFNESS MATRIX

Displacement based formulation of a Timoshenko beam utilizes a parabolic assumption of displacement variation. Such an assumption is approximate even for linear elastic material having prismatic cross sections. Hence, displacement based formulation does not provide exact closed form solutions for the deformation of a curved tapered beam.

Thus, the procedure presented by Fédération internationale du béton Task Group 4.4 [2] for force based formulation of a Timoshenko element is adopted for the case of a 2-D curved tapered Timoshenko beam element. The element is formulated referring a basic system, which is shown in Figure 1.


**Figure 1: Basic system with nodal displacements and nodal forces**

In usual notation, nodal displacement vector $\boldsymbol{U}$, nodal force vector $\boldsymbol{P}$, section force vector $\boldsymbol{\sigma}(x)$, section deformation vector $\boldsymbol{\epsilon}(x)$ and section flexibility matrix $\mathbf{f}$(x) are given in Eq. (1), (2), (3), (4), (5) respectively for a Timoshenko element. Axial force, bending moment, shear force, axial deformation of the reference axis, curvature and shear deformation at a distance x along the element are denoted by N($x$), M($x$), V($x$), $\varepsilon_0$ ($x$), κ($x$) and γ($x$) respectively.

$$\boldsymbol{U} = [\theta_1\ \theta_2\ u]^T \tag{1}$$

$$\boldsymbol{P} = [M_1\ M_2\ N]^T \tag{2}$$

$$\boldsymbol{\sigma}(x) = [N(x)\ \ M(x)\ \ V(x)]^T \tag{3}$$

$$\boldsymbol{\epsilon}(x) = [\varepsilon_0(x)\ \ \kappa(x)\ \ \gamma(x)]^T \tag{4}$$

$$\mathbf{f}(\mathrm{x}) = \begin{pmatrix} \frac{1}{EA(x)} & 0 & 0 \\ 0 & \frac{1}{EI(x)} & 0 \\ 0 & 0 & \frac{1}{GA_s(x)} \end{pmatrix} \tag{5}$$

Force interpolation functions $\mathbf{N_p}$(x) are selected such that equilibrium is satisfied in the strong form at any interior point, without the need for an approximation for displacements, stresses or strains. Equilibrium is expressed as in Eq. (6) for the Timoshenko beam. $\mathbf{L_\sigma}$ is the force differential operator.

$$\mathbf{L}_{\boldsymbol{\sigma}}^T\boldsymbol{\sigma}(\mathrm{x}) = \mathbf{0} \tag{6}$$

$$\mathbf{L}_{\boldsymbol{\sigma}}^T = \begin{pmatrix} \frac{d}{dx} & 0 & 0 \\ 0 & \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d}{dx} & -1 \end{pmatrix} \tag{7}$$

Force interpolation functions are obtained solving Eq. (6).

$$\mathbf{N_p}(\mathrm{x}) = \begin{pmatrix} 0 & 0 & 1 \\ \left(\frac{x}{L} - 1\right) & \frac{x}{L} & 0 \\ \frac{1}{L} & \frac{1}{L} & 0 \end{pmatrix} \tag{8}$$

Section forces are expressed as a function of nodal forces in Eq. (9).

$$\boldsymbol{\sigma}(\mathrm{x}) = \mathbf{N_p}(\mathrm{x})\boldsymbol{P} \tag{9}$$

Section constitutive law is given in Eq. (10).

$$\boldsymbol{\epsilon}(x) = \mathbf{f}(\mathrm{x})\boldsymbol{\sigma}(x) \tag{10}$$

Compatibility is enforced in the weak form through principle of virtual forces, as expressed in Eq. (11).

$$\delta\boldsymbol{P}^T\boldsymbol{U} = \int_0^L \delta\,\boldsymbol{\sigma}^T(x)\boldsymbol{\epsilon}(x)\mathrm{dx} \tag{11}$$

A relationship among U and P can be obtained combining Eq. (9), (10) and (11) and is given in Eq. (12), where F is the element flexibility matrix referring the basic system.

$$\boldsymbol{U} = \boldsymbol{F}\,\boldsymbol{P} \tag{12}$$

$$\boldsymbol{F} = \int_0^L \mathbf{N}_{\boldsymbol{p}}^{\boldsymbol{T}}(\boldsymbol{x})\,\mathbf{f}(\mathrm{x})\mathbf{N_p}(\mathrm{x})\,dx \tag{13}$$

Element stiffness matrix is formulated for a 2-D curved tapered Timoshenko beam element with rectangular cross section of uniform width $b$ and varying depth according to any smooth function *h(x).*

Section flexibility matrix for the said element is presented in Eq. (14), where $k$ denotes the Timoshenko coefficient for a rectangular section.

$$\mathbf{f}(x) = \begin{pmatrix} \frac{1}{Ebh(x)} & 0 & 0 \\ 0 & \frac{12}{Ebh(x)^3} & 0 \\ 0 & 0 & \frac{1}{Gkbh(x)} \end{pmatrix} \tag{14}$$

Element flexibility matrix $\boldsymbol{F}$ referring the basic system is computed by substituting in Eq. (13). Numerical integration should be performed in this step. The authors have used Gauss-Lobatto numerical integration procedure.

Element stiffness matrix $\boldsymbol{K}$ referring the basic system is computed by inverting $\boldsymbol{F}$. Formulation is performed for the basic system so that $\boldsymbol{F}$ is invertible.

Element stiffness matrix $\mathbf{K}_L$ and element force vector $\mathbf{P}_L$ referring the local coordinate system can be obtained by adding the rigid body modes and are given in Eq. (15) and Eq. (16) respectively. Rigid body transformation matrix is denoted by $\boldsymbol{\Gamma}_{RBM}$.

$$\mathbf{K}_L = \boldsymbol{\Gamma}_{RBM}^T\,\boldsymbol{K}\,\boldsymbol{\Gamma}_{RBM} \tag{15}$$

$$\mathbf{P}_L = \boldsymbol{\Gamma}_{RBM}^T\,\boldsymbol{P} \tag{16}$$

In the linear elastic case, referring the local coordinate system, element nodal displacement vector $\mathbf{U}_L$ can be computed as per Eq. (17).

$$\mathbf{U}_L = \mathbf{K}_L^{-1}\,\mathbf{P}_L \tag{17}$$

## 3. CASE STUDY

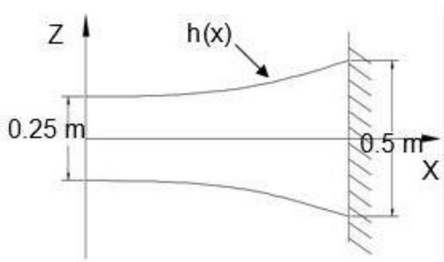


**Figure 2. Dimensions of the curved tapered beam**

Consider a curved tapered cantilever beam with rectangular cross section of uniform width and parabolic increase of depth along the element. Length (L) and width (b) are 2.0 m and 0.4 m respectively. Depth increases from the free end to the fixed far end of the element according to Eq. (18).

$$h(x) = 0.03125x^2 + 0.125 \qquad (18)$$

Dimensions of the starting section and end section are as given in Figure 2. Mechanical properties of reinforced concrete material are modulus of elasticity (E) of $33\times10^6$ kN/m$^2$ and Poisson's ratio (υ) of 0.15. A concentrated load (P) of 10 kN is applied at the free end. It is important to note that this study is conducted neglecting the self-weight of the element.

Displacement of the free end obtained from the proposed force based model and discretized displacement based models was examined.

## 4. RESULTS

Figure (3) illustrates that displacement based models require very fine discretization using at least 32 elements for the convergence of the displacement of the free end.

On the other hand, the proposed force based model provides an exact answer under the assumptions of Timoshenko beam theory, using just one element. The accuracy of the results is only conditioned by the Gauss-Lobatto numerical integration procedure.

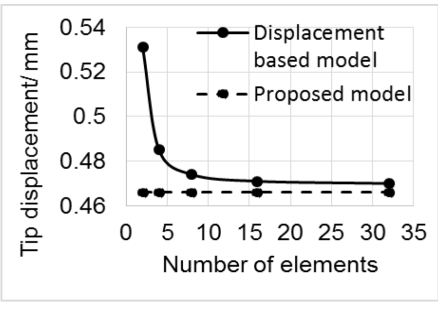


**Figure 3. Tip displacement variation with respect to number of elements**

This highlights the fact that the proposed force based model provides accurate results for the linear static analysis of 2-D curved tapered Timoshenko beam members.

Furthermore, it is evident that the proposed force based model is computationally much more economical than the discretized displacement methods when it comes to the linear static analysis 2-D curved tapered members.

## 5. CONCLUSION

A force based formulation was adopted for the linear static analysis of 2-D curved tapered Timoshenko beam elements with uniform width and variable height. This formulation permits 2-D modelling of any structure consisting of such members and provides accurate results for the linear static analysis, conditioned only by the numerical integration procedure used and the used beam theory assumptions.

The superior computational efficiency and the accuracy of results of the proposed force based model when compared with discretized displacement based methods were illustrated using an example.

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## 6. REFERENCES

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