FREE VIBRATION CHARACTERISTICS OF A 2D TAPERED BEAM USING FORCE-BASED FINITE ELEMENT FORMULATION

M. A. K. M. Dharmasiri[1], K. K. Wijesundara[1], A . J. Dammika, P. B. R. Dissanayake

[1] Department of Civil Engineering, Faculty of Engineering, University of Peradeniys, Peradeniya, Sri Lanka.
Email: makmddharmasiri@ymail.com

ABSTRACT

For modal analysis most widely used approach is using displacement based finite elements to arrive at approximate solutions. Since, an assumed displacement field is used in the displacement based formulation, tapered beams cannot be modelled using a typical displacement based finite element. Either case specific displacement based element or otherwise fine discretization is required to arrive at reasonable results. Force based finite element formulation uses satisfaction of equilibrium of forces at an internal point. In this study, free vibration characteristics of a tapered cantilever beam obtained by force based finite element formulation for the fundamental flexural mode of vibration is compared against results from a finely discretised displacement based finite element formulation model. To take account of distribution of mass and stiffness, consistent mass matrix is adapted. From the results, it is clear that force based finite element formulation gives better results efficiently with a single element and the discretised displacement based model results converge to that of force-based approach.

***Key words*:** Force-based formulation, Consistent mass matrix, Modal analysis, Tapered beam

1. INTRODUCTION

Civil engineering structures mostly constitute of members that are uniform in geometry throughout its length with the exception of structures having architectural features to enhance aesthetics or as stated by Molins *et. al.* 1998 [1] in ancient structures as part of the structure such as arches. Such constitute elements with complex geometries with non-uniform cross section. The modal analysis is used to validate numerical models, to identify damages, evaluate the life span of structures, etc. Therefore, in such a case it is important to model the structure with least number of degrees of freedom for computational efficiency enabling sustainability. However, coarser the discretization is, the results tend to be more approximate.

Usually, for static and modal analysis of elements with complex geometries the model is refined by using elements with uniform cross section with the section changing along the particular member [1] until a particular convergence criterion is met. Generally, displacement based finite elements are used in such cases.

In the Timoshenko beam displacement based finite element, the beam element that has to be used is three node. However, force-based finite element formulation does not use such crude hypotheses and disadvantages; instead, interpolation functions use satisfaction of equilibrium of forces at a point.

The stiffness matrix and consistent mass matrix based on force-based finite element formulation approach use the distribution of mass and stiffness without any underlying assumptions. Spacone *et.al.* 1996 [2] has used force-based finite element formulation for nonlinear analysis of RC frames. Molins *et. al.* 1998; Soydas and Saritas 2015 [1,3] have formulated consistent mass matrix based on force-based finite element formulation.

The study involves modal analysis of a tapered cantilever beam in 2D. A comparison is made at the end with a numerical model based on displacement-based finite element formulation.

2. METHODOLOGY

2.1. Displacement base finite element formulation

In displacement-based finite element formulation, an assumed displacement field is used to derive the stiffness matrix and consistent mass matrix. The displacement based formulation is approximate, even for a prismatic section since the parabolic displacement fields assumed for Timoshenko beam do not satisfy the beam governing differential equations. Therefore, in order to capture appropriate approximation for modal analysis a more finely refined discretized model is required. Otherwise case specific displacement fields suitable to model a tapered beam have to be identified.

2.2. Force Based Stiffness Matrix

Force-based finite element formulation relies on the satisfaction of equilibrium of forces at an internal point. No crude hypotheses for displacement fields are employed. The stiffness matrix is derived for a tapered beam having the degrees of freedom depicted in Figure 1.

Figure 1: Section forces and degrees of freedom associated with the tapered beam considered

Figure 2: Basic system with nodal displacements

The basic system used in the formulation is depicted in Figure 2. If $\boldsymbol{q_L}$ denotes the nodal forces at the far end and ***s(x)*** the section forces, they can be related by the shape function matrix, $\mathbf{N_p(x,L)}$

$$\boldsymbol{q_L} = [N\ V_2\ M_2]^T \quad \textbf{(1)}$$

$$\boldsymbol{s(x)} = [N(x)\ \ M(x)\ \ V(x)]^T \quad \textbf{(2)}$$

$$\boldsymbol{s(x)} = \mathbf{N_p}(x,L)\boldsymbol{q_L} \quad \textbf{(3)}$$

where N(x) is the axial force, M(x) is the bending moment and V(x) is the shear force at a distance x from the left end.

Interpolation functions are selected such that the equilibrium is satisfied in the strong form. Hence, the shape function takes the form of

$$\mathbf{N_p}(x,L) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & L-x & 1 \end{pmatrix} \quad \textbf{(4)}$$

The section deformations vector ***e(x,),***

$$\boldsymbol{e(x)}, = [\boldsymbol{\varepsilon_0(x)}\ \ \boldsymbol{\kappa(x)}\ \ \boldsymbol{\gamma(x)}]^T \quad \textbf{(5)}$$

where $\varepsilon_0(x)$ is the axial deformation of the reference axis, $\kappa(x)$ is the curvature about y axis and $\gamma(x)$ is the shear deformation about y axis, can be related to section forces ***s(x)*** by section flexibility matrix **f(x),**

$$\boldsymbol{e(x)} = \mathbf{f(x)}\boldsymbol{s(x)} \quad (6)$$

$$\mathbf{f}(\mathrm{x}) = \begin{pmatrix} EA(x) & EQ(x) & 0 \\ 0 & EI(x) & 0 \\ EQ(x) & 0 & GA_s(x) \end{pmatrix}^{-1} \quad \textbf{(7)}$$

Nodal displacements can be related to the displacements of the beam by,

$$\boldsymbol{u_L} = \boldsymbol{N_p^T(0,L)u_0} + \int_0^L \boldsymbol{N_p^T(x,L)\ e(x)\,dx} \quad \textbf{(8)}$$

where $\boldsymbol{u_L}$ and $\boldsymbol{u_0}$are nodal displacements at the far end and closer end respectively. It is apparent that element deformations vector $\boldsymbol{v}$ is,

$$\boldsymbol{v} = \boldsymbol{u_L} - \boldsymbol{N_p^T(0,L)u_0} = \int_0^L \boldsymbol{N_p^T(x,L)\ e(x)\,dx} \quad \textbf{(9)}$$

Hence, element flexibility matrix $\mathbf{F}_c$ (Soydas & Saritas, 2015) can be obtained,

$$\boldsymbol{v} = \mathrm{F}_c\boldsymbol{q_L} \quad \textbf{(10)}$$

$$\mathrm{F}_c = \int_0^L \mathbf{N_p^T(x,L)\ f(x)N_p(x,L)}\,\boldsymbol{dx} \quad \textbf{(11)}$$

Accordingly, nodal forces at far end, $\boldsymbol{q_L}$ can be expressed as,

$$\boldsymbol{q_L} = -\mathrm{F}_c^{-1}\mathrm{N}_c^T(\mathbf{0},\boldsymbol{L})\boldsymbol{u_0} + \mathrm{F}_c^{-1}\boldsymbol{u_L} \quad \textbf{(12)}$$

Then, nodal forces at closer end, $\boldsymbol{q_0}$ are,

$$\boldsymbol{q_0} = \boldsymbol{N_p^T(0,L)}\left(-\boldsymbol{F_C^{-1}N_p^T(0,L)u_0} + \boldsymbol{F_C^{-1}u_L}\right) \quad \textbf{(13)}$$

Results from Eq. 12 and 13 can be summarized in the following form,

$$\begin{pmatrix} \boldsymbol{q_0} \\ \boldsymbol{q_L} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{N_p^T(0,L)F_C^{-1}N_p^T(0,L)} & \boldsymbol{N_p^T(0,L)F_C^{-1}} \\ -\mathbf{F}_c^{-1}\mathbf{N}_c^T(\mathbf{0},\boldsymbol{L}) & \mathbf{F}_c^{-1} \end{pmatrix}\begin{pmatrix} \boldsymbol{u_0} \\ \boldsymbol{u_L} \end{pmatrix} \quad \textbf{(14)}$$

It is apparent that this takes the form of

$$\boldsymbol{P} = \mathbf{K}\boldsymbol{u} \quad \textbf{(15)}$$

Where $\boldsymbol{P}$ is the nodal force vector, $\boldsymbol{u}$ is the nodal displacement vector and **K** is the element stiffness referring local coordinate system. Thus, element stiffness matrix could be obtained.

2.3. Force Based Consistent Mass Matrix

Following a similar procedure as in the stiffness matrix, interpolation function N(*x*,*L*) can be obtained (Molins *et. al.*, 1998; Soydas and Saritas, 2015)

$$\mathrm{N}(\boldsymbol{x},\boldsymbol{L}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & L-x & 1 \end{pmatrix} \quad \textbf{(16)}$$

For the interpolation function, $\mathbf{N}(x, L)$ flexibility matrix $\mathbf{F_c}$ can be obtained using Eq. (10). The partial flexibility matrix $\mathbf{f_p}(\mathrm{x})$ for the element can be calculated as

$$\mathbf{f}_p(\mathrm{x}) = \int_0^x \mathbf{N}_p^T(\zeta, x)\ \mathbf{f}(\mathrm{x})\mathbf{N_p}(\zeta, \mathrm{x})\,d\zeta \tag{17}$$

Section mass matrix, $\mathbf{m_s}$ can be defined as

$$\mathbf{m}_s(x) = \begin{bmatrix} \boldsymbol{A(x)} & \mathbf{0} & \boldsymbol{Q(x)} \\ \mathbf{0} & \boldsymbol{A(x)} & \mathbf{0} \\ \boldsymbol{Q(x)} & \mathbf{0} & \boldsymbol{I(x)} \end{bmatrix} \tag{18}$$

Accordingly, consistent mass matrix, **M** for the element with three DOFs per node can be formulated using four sub matrices using the method provided by Molins *et. al.* 1998 [1].

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}_{oo} & \mathbf{m}_{oL} \\ \mathbf{m}_{Lo} & \mathbf{m}_{LL} \end{bmatrix} \tag{19}$$

$$\mathbf{m}_{LL} = \mathbf{F}_c^{-1} \int_0^L \mathbf{N}^T(x, L)\,\mathbf{f}(\mathrm{x}) \left(\int_x^L \mathbf{N}^T(x, \zeta)\mathbf{m}_s(\zeta)\,\mathbf{f}_p(\zeta)\mathbf{F}_c^{-1} d\zeta \right) dx \tag{20}$$

$$\mathbf{m}_{oL} = \mathbf{m}_{Lo}^T = -\mathbf{N}(0, L)\mathbf{m}_{LL} + \int_0^L \mathbf{N}(0, x)\mathbf{m}_s(x)\,\mathbf{f}_p(\mathrm{x})\mathbf{F}_c^{-1} dx \tag{21}$$

$$\mathbf{m}_{oo} = -\mathbf{N}(0, L)\mathbf{m}_{L0} + \int_0^L \mathbf{N}(0, x)\mathbf{m}_s(x) \times \left(\mathbf{N}^T(0, x) - \mathbf{f}_p(\mathrm{x})\mathbf{F}_c^{-1}\mathbf{N}^T(0, L)\right) dx \tag{22}$$

2.4. Modal Analysis

For undamped free vibration response of a structural system,

$$\mathbf{M}\ddot{\boldsymbol{U}} + \mathbf{K}\boldsymbol{U} = \mathbf{0} \tag{23}$$

Where **M** is the mass matrix and **K** is the stiffness matrix formulated using FBF approach. By solving the eigenvalue problem resulting from Eq. (20), natural frequencies and mode shapes can be obtained.

3. CASE STUDY

The dimensions of the tapered cantilever beam with rectangular cross section of uniform width and parabolic increase of depth along the element is shown in Figure 3. Length (L) and width (b) are 2.0 m and 0.4 m respectively. Depth decreases from fixed end to the free end of the element according to Eq. (21)

$$h(x) = 0.03125x^2 + 0.125 \tag{24}$$

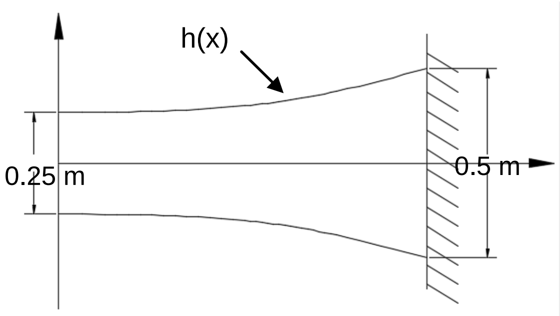

Figure 3: Dimensions of the tapered beam

Mechanical properties of the reinforced concrete material are Young's modulus (E) 33×10^6 kN/m^2 and Poisson's ratio (υ) 0.15.

4. RESULTS

Modal analysis is performed on the tapered cantilever beam by using force-based finite element formulation approach and a comparison is made with results from displacement-based finite element formulation approach model results

The natural frequencies obtained by solving the eigenvalue problem resulting from Eq. (20), from the single Timoshenko force-based element correspond to the fundamental modes of flexure, shearing and axial since only these three degrees of freedom were involved in the formulation.

The DBF model is discretized in several stages 2, 4, 8, 16 and 32 elements. It is discretized such that each element is of equal length and element depth decreases from fixed end to the near end gradually.

As shown in Figure 4, for the first flexure mode results seem to converge with the percentage difference between results for 16 and 32 elements being 0.31%. The natural frequency for fundamental flexure mode obtained by force-based approach using a single element is 71.5 Hz whereas 69.9 Hz is given by numerical model with 32 elements, the percentage difference being 2.2%.

A similar trend is observed for first axial mode as results seems to converge but at a higher frequency than of force-based element (Figure 5). The deviation could be attributed to the numerical method used in extracting modal parameters from the eigenvalue problem in the case of discretization in DBF model.

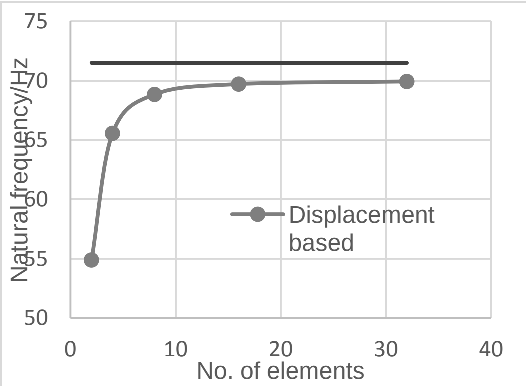

Figure 4: Natural frequency vs no. of elements for first flexure mode

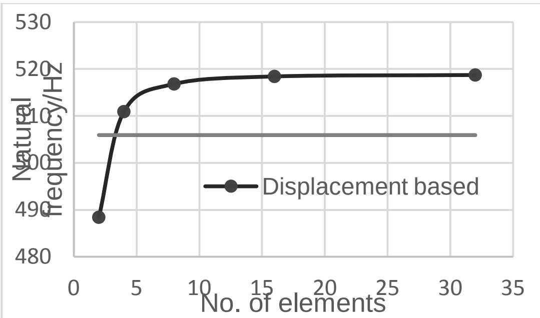

Figure 5: Natural frequency vs no of elements for first axial mode

5. CONCLUSION

From the results, it is evident that force- based finite element formulation approach gives exact results for natural frequencies using a single element for a curved tapered cantilever beam without any discretization. Therefore, it can be concluded that force-based approach is more efficient and accurate with respect to displacement-based formulation where discretization or use of case specific interpolation functions are required to arrive at results with satisfactory degree of approximation.

6. REFERENCES

[1] C. Molins, P. Roca,and A.H. Barbat, "*Flexibility-based linear dynamic analysis of complex structures with curved-3D members*", Earthquake Engineering and Structural Dynamics, vol. 27 no.7, pp.731-747, 1998.

[2] E. Spacone, F.C. Filippou, and F.F Taucer, *"Fibre beam-column model for non-linear analysis of R/C frames: Part I. Formulation"*, Earthquake engineering and structural dynamics, vol.25, pp. 711–726.1996.

[3] O. Soydas, and A. Saritas, *"Free vibration characteristics of a 3d mixed formulation beam element with force-based consistent mass matrix*", Journal of Vibration and Control, 2015. 1077546315619263